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ON THE BIEBERBACH AND KOEBE CONSTANTS FOR SECTOR DOMAINS AND SECTOR DISKS

BY

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ABSTRACT

We discuss domain constants related to the classical Bieberbach and Koebe theorems. We find a class of simply connected domains for which the product of these constants behave like extremal domain and gives a better result on Osgood's inequalities.

1. Introduction

A domain D in the complex plane $\mathbf{C}$ is hyperbolic if its complement contains at least two points. For a point w in D, we denote by $\lambda_D(w)$ the hyperbolic metric of D at w and by $\delta_D(w)$ the infimum Euclidean distance from w to the boundary of D. We define, as in [M],

$$C(D) = \inf\{\lambda_D(w)\delta_D(w) \colon w \in D\}$$

and, as in [HM],

$$\eta(D) = \frac{1}{2}\sup\left\{\frac{|\nabla \log \lambda_D(w)|}{\lambda_D(w)} \colon w \in D\right\}.$$

We call, as in [R1], [R2] and [TR], $\eta(D)$ the **Bieberbach constant** and $C(D)$ the **Koebe constant** for a hyperbolic domain D. We point out, as in Pommerenke [P2], that a hyperbolic domain is uniformly perfect if and only if $C(D) > 0$.

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We focus on Osgood's inequalities [O, Theorem 5] which state that, if D is a hyperbolic domain, then

$$\frac{1}{2} \leq \eta(D)C(D) \leq 1 \tag{1}$$

The upper bound appears not to be sharp for simply connected domains. For example, if D is a convex domain then $\eta(D)C(D) = 1/2$. If D is a slit plane then $\eta(D)C(D) = 1/2$. Furthermore, Rugeihyamu [R1] and [R2] proved that if D is a spike domain or a slit disk, then $\eta(D)C(D) = 1/2$.

In this paper we obtain another class of simply connected domains with $\eta(D)C(D) = 1/2$. We also study a second class of simply connected domains, which we believe has the same property. First we need the following definitions:

Definition 1: Set $0 < \rho \leq 4$. A domain D_ρ is said to be a **sector domain** if

$$D_\rho = \{w\colon |\arg w| < \rho\pi/4\}.$$

Note that if $\rho = 2$, then D_ρ becomes the right half plane. If $\rho = 4$, then D_ρ becomes the slit plane which omits the negative real axis.

Definition 2: Set $0 < \rho \leq 4$. A domain S_ρ is said to be a **sector disk** if

$$S_\rho = \{w\colon |\arg w| < \rho\pi/4 \text{ and } |w| < r\},$$

where r is a finite positive real number.

S_4 is the slit disk given by $S_4 = D(0, r) \setminus [-r, 0]$.

Definition 3: Let $\{D_n\}$ be a sequence of domains. We define the **pre-kernel** of $\{D_n\}$ to be the set

$$\{w\colon \text{ there is a positive } r \text{ so that } \overline{D(w, r)} \subseteq D_n \text{ for all sufficiently large } n\}.$$

If z_0 lies in the pre-kernel, we define the **kernel** of $\{D_n\}$ **with respect to** z_0 to be the component of the pre-kernel that contains z_0. If the pre-kernel does not contain z_0, then the sequence $\{D_n\}$ is said not to have a kernel with respect to z_0. Furthermore, we say that a sequence of domains $\{D_n\}$ converges to D with respect to z_0 if z_0 lies in the pre-kernel of $\{D_n\}$ and each subsequence of $\{D_n\}$ has the same kernel D with respect to z_0. This is indicated by the notation $D_n \to D$ as $n \to \infty$ with respect to z_0.

We also use the following Carathéodory Kernel Convergence Theorem [C, p. 85–90], THEOREM A: *Suppose that for each $n \geq 1$, f_n is a conformal map of the unit disk U onto a domain D_n with $f_n(0) = z_0$ and $f_n'(0) > 0$. Then the sequence of conformal maps $\{f_n\}$ converges uniformly on each compact subset of U, say to f, if and only if the sequence of domains $\{D_n\}$ converges to a domain D with respect to z_0. In the case of convergence, f is a conformal map of the unit disk U onto D with $f(0) = z_0$ and $f'(0) > 0$.*

To motivate our work, let $0 < \rho \leq 4$ and w be a point in D_ρ. Set $w = re^{i\theta}$, then

$$\frac{|\nabla \log \lambda_{D_\rho}(w)|}{\lambda_{D_\rho}(w)} = \sqrt{4 + (\rho^2 - 4)\cos(2\theta/\rho)}.$$

We deduce the result proved by [P1, p. 117], [Y, p. 173] and [R2, p. 62], that is, if $2 \leq \rho \leq 4$, then

$$\eta(D) = \rho/2. \tag{2}$$

Our first Theorem is

THEOREM 1: *Suppose that D_ρ is a sector domain and that $2 \leq \rho \leq 4$. Then $C(D_\rho) = 1/\rho$.*

To our second result we prove

THEOREM 2: *Suppose that S_ρ is a sector disk and that $2 \leq \rho \leq 4$. Then $C(S_\rho) = 1/\rho$.*

Rugeihyamu [R1] and [R2] proved that for a slit disk $C(S_4) = 1/4$ and $\eta(S_4) = 2$. Also note that if $0 < \rho \leq 2$, then S_ρ is a convex domain with $C(S_\rho) = 1/2$ and $\eta(S_\rho) = 1$. Our last result gives a better estimate in (1) as ρ approaches 4.

THEOREM 3: *Suppose that S_ρ is a sector disk and that $2 \leq \rho \leq 4$. Then*

$$\rho/2 \leq \eta(S_\rho) \leq 2.$$

2. Main results

For a sector domain D_ρ one has

$$\frac{|\nabla \log \lambda_{D_\rho}(x)|}{2\lambda_{D_\rho}(x)} = \frac{\rho}{2} \quad \text{for each } x > 0.$$

If $2 \leq \rho \leq 4$, then the extremal value for the Bieberbach constant $\eta(D_\rho)$ occurs along the positive real axis. The extremal value for the Koebe constant $C(D_\rho)$ also occurs along the positive real axis.

Proof of Theorem 1: Let w be a point in D_ρ. The function $f_\rho = w^{\rho/2}$ is a conformal map of the right half plane onto the domain D_ρ. By the pullback metric of the right half plane, the hyperbolic metric of D_ρ is given by

$$\lambda_{D_\rho}(w) = \frac{1}{\rho|w^{(\rho-2)/\rho}\,\mathrm{Re}(w^{2/\rho})|}, \quad w \in D_\rho.$$

In particular, if $w = x$ is a point which lies on the positive real axis of D_ρ, then

$$\lambda_{D_\rho}(x) = 1/\rho x \quad \text{for each } x > 0.$$

The origin is the closest point on the boundary of D_ρ to the point x. Thus, $\delta_{D_\rho}(x) = x$ and

$$\lambda_{D_\rho}(x)\delta_{D_\rho}(x) = 1/\rho \quad \text{for each } x \text{ in } D_\rho.$$

It follows that $C(D_\rho) \leq 1/\rho$. On the other hand, by Osgood's inequality (1) and (2), we have

$$C(D_\rho) \geq 1/2\eta(D_\rho) = 1/\rho.$$

Combining the inequalities gives the required result. ∎

Proof of Theorem 2: By scale invarinace we may assume that $r = 2$. We set $S_\rho^n = nS_\rho = \{w\colon w/n \in S_\rho\}$ for all $n \geq 1$. Since the Koebe constant is invariant under rotation, translation and scaling of a domain it follows that

$$\lambda_{S_\rho}(w/n)\delta_{S_\rho}(w/n) = \lambda_{S_\rho^n(w)}\delta_{S_\rho^n}(w), \quad w \in S_\rho.$$

In particular, if $w = 1$ then

$$\lambda_{S_\rho}(1/n)\delta_{S_\rho}(1/n) = \lambda_{S_\rho^n}(1)\delta_{S_\rho^n}(1). \tag{3}$$

Next we consider the kernel of the sequence $\{S_\rho^n\}$ with respect to 1. Since the sequence $\{S_\rho^n\}$ is increasing with $\bigcup\{S_\rho^n\} = D_\rho$ then, by definition, the sector domain D_ρ is the kernel of $\{S_\rho^n\}$ with respect to 1.

Let f_n be the conformal map of U onto S_ρ^n normalised so that $f_n(0) = 1$ and $f_n'(0) > 0$ for all $n \geq 1$. By the Carathéodory Kernel Convergence Theorem the functions $f_n(z)$ converge uniformly in each compact subset of U to $f_\rho(z)$, where f_ρ is the conformal map of U onto D_ρ given by

$$f_\rho(z) = \left(\frac{1+z}{1-z}\right)^{\rho/2}.$$

By a standard argument, $f_n'(z) \to f_\rho'(z)$ uniformly in each compact subset of U. In particular, $f_n'(0) \to f_\rho'(0) = \rho$ as $n \to \infty$ and hence

$$\lim_{n\to\infty} \frac{1}{|f_n'(0)|} = \frac{1}{\rho}. \tag{4}$$

By the pullback of the hyperbolic metric, $\lambda_{S_\rho^n}(f_n(z))|f_n'(z)| = \lambda_U(z)$ for each z in U. In particular, $\lambda_{S_\rho^n}(f_n(0)) = 1/|f_n'(0)|$. Since $f_n(0) = 1$ for all $n \geq 1$, then by (4) we have

$$\lim_{n\to\infty} \lambda_{S_\rho^n}(1) = 1/\rho. \tag{5}$$

Let x be a point in $(0,1]$. Then the origin is the closest point on the boundary of S_ρ^n to x. Thus, $\delta_{S_\rho^n}(x) = x$ for all $n \geq 1$. In particular, if $x = 1$ then $\delta_{S_\rho^n}(1) = 1$ for all $n \geq 1$. We combine this result with (5) to obtain

$$\lim_{n\to\infty} \lambda_{S_\rho^n}(1)\delta_{S_\rho^n}(1) = 1/\rho. \tag{6}$$

Combining (3) and (6) gives $\lim_{n\to\infty} \lambda_{S_\rho}(1/n)\delta_{S_\rho}(1/n) = 1/\rho$. Thus $C(S_\rho) \leq 1/\rho$.

Next we show that $C(S_\rho) \geq 1/\rho$. Set $E_0 = \{w_0 \in \partial S_\rho\colon |w_0| < 2\}$ and $E_1 = \{w_0 \in \partial S_\rho\colon |w_0| = 2\}$. Then we divide the points in S_ρ into two regions F_0 and F_1 where $F_0 = \{w \in S_\rho\colon \delta_{S_\rho}(w) = \operatorname{dist}(w, E_0)\}$ and $F_1 = \{w \in S_\rho\colon \delta_{S_\rho}(w) = \operatorname{dist}(w, E_1)\}$. It follows that

$$\begin{aligned} C(S_\rho) &= \inf\{\lambda_{S_\rho}(w)\delta_{S_\rho}(w)\colon w \in S_\rho\} \\ &= \min\{\inf\{\lambda_{S_\rho}(w)\delta_{S_\rho}(w)\colon w \in F_0\}, \inf\{\lambda_{S_\rho}(w)\delta_{S_\rho}(w)\colon w \in F_1\}\}. \end{aligned}$$

Since $S_\rho \subset D_\rho$ and $S_\rho \subset D = D(0,2)$, then by the monotonicity property of the hyperbolic metric we have $\lambda_{D_\rho}(w) \leq \lambda_{S_\rho}(w)$ and $\lambda_D(w) \leq \lambda_{S_\rho}(w)$ for each $w \in S_\rho$. In particular, if w is a point in F_0, then $\delta_{S_\rho}(w) = \delta_{D_\rho}(w) = \operatorname{dist}(w, E_0)$ and so

$$\lambda_{D_\rho}(w)\delta_{D_\rho}(w) \leq \lambda_{S_\rho}(w)\delta_{S_\rho}(w), \quad \text{for each } w \in F_0. \tag{7}$$

If w is a point in F_1, then $\delta_{S_\rho}(w) = \delta_D(w) = \operatorname{dist}(w, E_1)$ and

$$\lambda_D(w)\delta_D(w) \leq \lambda_{S_\rho}(w)\delta_{S_\rho}(w), \quad \text{for each } w \in F_1. \tag{8}$$

It follows from (7) and (8) that

$$\begin{aligned}
C(S_\rho) &\geq \min\{\inf\{\lambda_{D_\rho}(w)\delta_{D_\rho}(w)\colon w \in F_0\}, \inf\{\lambda_D(w)\delta_D(w)\colon w \in F_1\}\} \\
&\geq \min\{\inf\{\lambda_{D_\rho}(w)\delta_{D_\rho}(w)\colon w \in D_\rho\}, \inf\{\lambda_D(w)\delta_D(w)\colon w \in D\}\} \\
&= \min\{C(D_\rho), C(D)\} \\
&= \min\{C(D_\rho), 1/2\} \\
&= \min\{1/\rho, 1/2\} \quad \text{by Theorem 1} \\
&= 1/\rho.
\end{aligned}$$

Thus $C(S_\rho) \geq 1/\rho$. Combining the two inequalities gives $C(S_\rho) = 1/\rho$. ∎

Proof of Theorem 3: The lower bound is obtained by combining (1) and Theorem 2. The upper bounds follows from the fact that a domain S_ρ is simply connected. ∎

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